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N=2 Supersymmetric RG Flows and the IIB Dilaton

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We show that there is a non-trivial relationship between the dilaton of IIB supergravity, and the coset of scalar fields in five-dimensional, gauged $\mathcal{N} = 8$ supergravity. This has important consequences for the running of the gauge coupling in massive perturbations of the AdS/CFT correspondence. We conjecture an exact analytic expression for the ten-dimensional dilaton in terms of five-dimensional quantities, and we test this conjecture. Specifically, we construct a family of solutions to IIB supergravity that preserve half of the supersymmetries, and are lifts of supersymmetric flows in five-dimensional, gauged $\mathcal{N} = 8$ supergravity. Via the AdS/CFT correspondence these flows correspond to softly broken $\mathcal{N} = 4$, large N Yang-Mills theory on part of the Coulomb branch of $\mathcal{N} = 2$ supersymmetric Yang-Mills. Our solutions involve non-trivial backgrounds for all the tensor gauge fields as well as for the dilaton and axion.

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1. Introduction

It has been evident over the last year that five-dimensional supergravity theories are very powerful tools in the study of the AdS/CFT correspondence [1,2,3]. In particular, gauged $\mathcal{N} = 8$ supergravity in five dimensions [4,5] describes $\mathcal{N} = 4$ Yang-Mills theory in the large N limit under perturbations that involve fermion or scalar bilinear operators [2,3,6,7,8]. What has been less evident is exactly how the five-dimensional solutions are lifted to ten-dimensional solutions. An example of such a lift was given in [9] for the non-trivial, supersymmetric critical point of [8], however, as yet, no non-trivial lifts of massive five-dimensional flows have been obtained. One of the purposes of this paper is to give the exact ten-dimensional solution for the five-dimensional $\mathcal{N} = 2$ supersymmetric flows.

The other, and more far-reaching purpose of this paper is to solve a beautiful subtlety in consistent truncation: a subtlety that has significant consequences for the field theory side of the AdS/CFT correspondence. Specifically, there is an $SL(2, \mathbb{R})$ invariance of the five-dimensional gauged supergravity theory, and perturbatively, the coset, $SL(2, \mathbb{R})/SO(2)$ corresponds to the ten-dimensional dilaton and axion. Combining these facts, it is a natural conflation to assume that this is always true: *i.e.* that the $SL(2, \mathbb{R})$ invariance in the five-dimensional theory “sweeps out” the ten-dimensional dilaton/axion coset. This turns out to be false, and indeed false in a very interesting way.

The scalars of $\mathcal{N} = 8$ supergravity are described by the coset $E_{6(6)}/USp(8)$. In terms of $SO(6)$ representations, the non-compact generators constitute the $\mathbf{20}' \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1} \oplus \mathbf{1}$. The two singlets are dual to the gauge coupling and theta-angle, while the $\mathbf{20}' \oplus \mathbf{10} \oplus \overline{\mathbf{10}}$ are respectively dual to the Yang-Mills scalar and fermion bilinears:

$$\text{Tr}(X^A X^B) - \frac{1}{6} \delta^{AB} \text{Tr}(X^C X^C), \quad \text{Tr}(\lambda^i \lambda^j), \quad \text{Tr}(\bar{\lambda}^{\bar{i}} \bar{\lambda}^{\bar{j}}). \quad (1.1)$$

The subgroup $SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) \subset E_{6(6)}$ describes the Yang-Mills theory on the Coulomb branch [10,11], or under purely scalar mass perturbations. In this sector the $SL(2, \mathbb{R})$ factor can indeed be identified with the ten-dimensional dilaton/axion coset. However, for more general $E_{6(6)}$ matrices, *i.e.* when fermion masses or vevs are non-zero it turns out that the relationship is far from trivial. Indeed, in this paper we conjecture that the $SL(2, \mathbb{R})$ matrix, S , that parametrizes the ten-dimensional IIB dilaton/axion is related to the scalar $E_{6(6)}$ matrix, \mathcal{V} , of five-dimensional supergravity via:

$$\Delta^{-\frac{4}{3}} (S S^T)^{\alpha\beta} = \text{const} \times \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \mathcal{V}_{I\gamma}{}^{ab} \mathcal{V}_{J\delta}{}^{cd} x^I x^J \Omega_{ac} \Omega_{bd}. \quad (1.2)$$

In this equation x^I are the cartesian coordinates on the compactification 5-sphere: $\sum_I (x^I)^2 = 1$, and Δ is related to the determinant of the internal metric. One may also define Δ by taking the determinant of both sides of (1.2) and using the unimodularity of S .

The five-dimensional scalar potential is invariant under $SL(2, \mathbb{R})$, and this is broken to $SL(2, \mathbb{Z})$ in the string theory. The non-compact generators of this $SL(2, \mathbb{R})$ are thus naturally identified with the $\mathcal{N} = 4$ gauge coupling. Equation (1.2) thus shows that the gauge coupling and theta-angle on the branes is a non-trivial, but exactly known function of the fermion and scalar masses and vevs, and of the $\mathcal{N} = 4$ gauge coupling. Indeed, most of the five-dimensional flows considered to date have constant values for the five-dimensional “dilaton and axion.” Equation (1.2) gives precisely the non-trivial running of the coupling for such flows, and presumably for $N = 1$ supersymmetric flows it should subsume an integrated version of the NSVZ exact beta-function [12].

We will examine some of these ideas in this paper, and test the conjecture (1.2) by considering the $\mathcal{N} = 2$ supersymmetric flow. In this flow, $\mathcal{N} = 4$ supersymmetric Yang-Mills is softly broken to $\mathcal{N} = 2$ by introducing a mass for the adjoint hypermultiplet. Gauged $\mathcal{N} = 8$ supergravity also allows us to probe a lowest mode of the Coulomb branch of the $\mathcal{N} = 2$ theory: there is a supergravity scalar that corresponds to turning Yang-Mills scalar vevs that, in the absence of the fermion mass, corresponds to spreading out the branes into a uniform disk distribution. For this flow, (1.2), in principle, gives a supergravity prediction for $\tilde{\tau}(\tau, m, u)$, where $\tilde{\tau}$ is the running $\mathcal{N} = 2$ coupling, τ is the $\mathcal{N} = 4$ coupling, m is the fermion mass and u is the non-trivial scalar vev. As we will see there are some subtleties yet to be understood.

We begin in section 2 by reviewing the five-dimensional description of the $\mathcal{N} = 2$ supersymmetric flow, and we compute the running of the dilaton predicted by (1.2). In section 3 we use the results of [8] to give the exact ten-dimensional metric for the flow, and then we examine the linearized Ansatz for the ten-dimensional 2-form fields. We then show that the $\mathcal{N} = 2$ flow must necessarily involve a running ten-dimensional dilaton. In section 4 we obtain the complete ten-dimensional solution, confirming our prediction of the dilaton/axion behaviour. Finally, in an appendix we give the consistent truncation argument that led us to the formula (1.2).

2. The $N = 2$ RG flow in five dimensions

The flow that preserves $\mathcal{N} = 2$ supersymmetry can be obtained from the superpotentials considered in [13]. We need to turn on the supergravity scalar fields dual to the operators:

$$\mathcal{O}_b = \sum_{j=1}^4 \text{Tr}(X^j X^j) - 2 \sum_{j=5}^6 \text{Tr}(X^j X^j), \quad \mathcal{O}_f = \text{Tr}(\lambda^3 \lambda^3 + \lambda^4 \lambda^4), \quad (2.1)$$

and to the complex conjugate operator, $\overline{\mathcal{O}}_f$. In the conventions of [13], the corresponding supergravity scalars are α and $\chi = \varphi_1 = \varphi_2$. On this subsector the tensor W_{ab} has two distinct eigenvalues, each with degeneracy 4. One of these two eigenvalues provides a superpotential for the flow:¹

$$W = -\frac{1}{\rho^2} - \frac{1}{2} \rho^4 \cosh(2\chi), \quad (2.2)$$

where, as usual, $\rho = e^\alpha$, and where the potential is given by:

$$\begin{aligned} \mathcal{P} &= -\frac{g^2}{4} \rho^{-4} - \frac{g^2}{2} \rho^2 \cosh(2\chi) + \frac{g^2}{16} \rho^8 \sinh^2(2\chi) \\ &= \frac{g^2}{48} \left(\frac{\partial W}{\partial \alpha} \right)^2 + \frac{g^2}{16} \left(\frac{\partial W}{\partial \chi} \right)^2 - \frac{g^2}{3} W^2. \end{aligned} \quad (2.3)$$

The kinetic term on this sector is: $-3(\partial\alpha)^2 - (\partial\chi)^2$.

Taking the flow metric to have the form

$$ds_{1,4}^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu - dr^2, \quad (2.4)$$

one then finds that supersymmetric flow equations are:

$$\begin{aligned} \frac{d\alpha}{dr} &= \frac{g}{12} \frac{\partial W}{\partial \alpha} = \frac{g}{6} \left(\frac{1}{\rho^2} - \rho^4 \cosh(2\chi) \right), \\ \frac{d\chi}{dr} &= \frac{g}{4} \frac{\partial W}{\partial \chi} = -\frac{g}{4} \rho^4 \sinh(2\chi), \end{aligned} \quad (2.5)$$

along with the auxilliary equation

$$\frac{dA}{dr} = -\frac{g}{3} W. \quad (2.6)$$

¹ Note that the eigenvalue that provides the superpotential for the flow considered here is not the same as the eigenvalue that provided the $\mathcal{N} = 1$ flow in [13].

The first step to solving the system (2.5),(2.6) is to write everything as a function of χ . Thus:

$$\frac{d\alpha}{d\chi} = -\frac{2}{3} \left(\frac{1}{\rho^6 \sinh(2\chi)} - \frac{\cosh(2\chi)}{\sinh(2\chi)} \right), \quad (2.7)$$

and

$$\frac{dA}{d\chi} = -\frac{2}{3} \left(\frac{2}{\rho^6 \sinh(2\chi)} + \frac{\cosh(2\chi)}{\sinh(2\chi)} \right). \quad (2.8)$$

First note that:

$$\frac{d}{d\chi} (A - 2\alpha) = -2 \frac{\cosh(2\chi)}{\sinh(2\chi)}. \quad (2.9)$$

This is trivially integrated with respect to χ , and it yields

$$e^A = k \frac{\rho^2}{\sinh(2\chi)}, \quad (2.10)$$

where k is a constant.

To integrate the equation for α , simply define $\beta = \alpha - \frac{1}{3} \log(\sinh(2\chi))$ and observe that

$$\frac{d\beta}{d\chi} = -\frac{2}{3} \frac{1}{\rho^6 \sinh(2\chi)} = -\frac{2}{3} \frac{e^{-6\beta}}{\sinh^3(2\chi)}.$$

It is also elementary to integrate this, and one then finds:

$$\begin{aligned} \rho^6 &= \cosh(2\chi) + \sinh^2(2\chi) \left[\gamma + \log \left(\frac{\sinh(\chi)}{\cosh(\chi)} \right) \right] \\ &= c + (c^2 - 1) \left[\gamma + \frac{1}{2} \log \left(\frac{c-1}{c+1} \right) \right], \end{aligned} \quad (2.11)$$

where $c = \cosh(2\chi)$ and γ is a constant of integration.

This formula has different asymptotics for γ positive, negative or zero. Since the superpotential has a manifest symmetry under $\chi \rightarrow -\chi$, we focus on $\chi > 0$: If γ is positive then $\alpha \sim \frac{2}{3}\chi + \frac{1}{6} \log(\frac{\gamma}{4})$ for large (positive) χ . If γ is negative then χ limits to a finite value as α goes to $-\infty$. If $\gamma = 0$ then we get the interesting ridge-line flow with $\alpha \sim -\frac{1}{3}\chi + \frac{1}{6} \log(\frac{4}{3})$ for large (positive) χ . Some of these flows are shown in Figure 1.

We claim that the choice, $\gamma = 0$ corresponds to the pure $\mathcal{N} = 2$ flow with vanishing scalar vev. The simplest argument for this claim is obtained from Figure 1. There are several obvious ridges in this figure. The ridges with $\chi = 0$ and α varying are two of the $\mathcal{N} = 4$ supersymmetric Coulomb branch flows identified in [11]. The two other ridges are equivalent under $\chi \rightarrow -\chi$ and are obtained by setting $\gamma = 0$. They correspond to massive supersymmetric flows, and there is only one such “preferred flow” namely the

pure $\mathcal{N} = 2$ flow with no scalar vev. One should note that the flow along the α -axis to the left corresponds to the Coulomb branch in which the branes spread out in a disk in the (X^5, X^6) directions, whereas the other direction corresponds to a brane distribution in the (X^1, X^2, X^3, X^4) directions. The moduli space that we seek makes the scalars (X^1, X^2, X^3, X^4) massive, but leaves us with the vevs in the (X^5, X^6) directions. Hence it is natural to identify the softly broken $\mathcal{N} = 4$ theory with the $\gamma = 0$ ridge and everything to the left of it. This is essentially the argument given in [14], where it was further argued that the flows to the right of the $\gamma = 0$ should be viewed as unphysical.

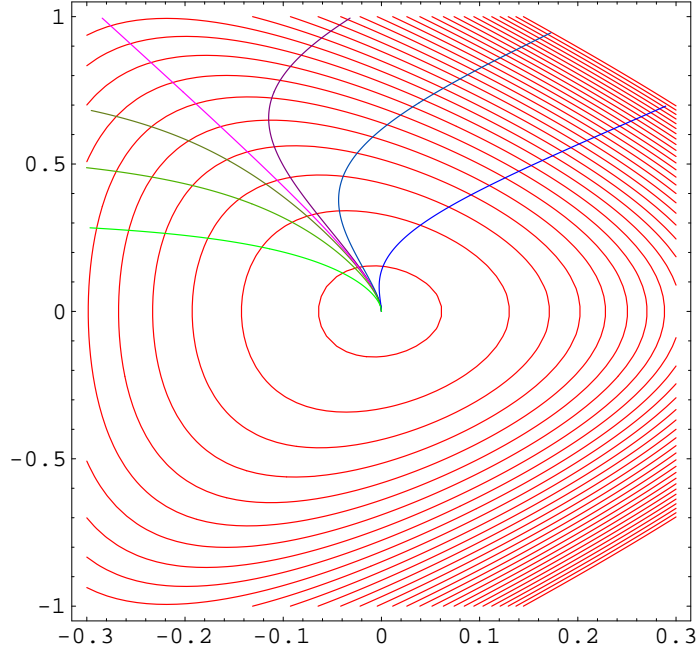


Fig. 1: Contours of the superpotential showing some of the steepest descent flows. The horizontal and vertical axis are α and χ , respectively. The middle, ridge-line flow has $\gamma = 0$ and the three flows to the left and right have $\gamma < 0$ and $\gamma > 0$, respectively.

3. The ten-dimensional solution

In this section we “lift” the $\mathcal{N} = 2$ flows to solutions of the chiral IIB supergravity in ten dimensions [15,16]. As we have already explained above, one should expect to find that all bosonic fields in ten dimensions are non-vanishing and as a result all bosonic field equations become nontrivial. Those equations consist of [15]:

- The Einstein equations:

$$R_{MN} = T_{MN}^{(1)} + T_{MN}^{(3)} + T_{MN}^{(5)}, \quad (3.1)$$

where the energy momentum tensors of the dilaton/axion field, B , the three index anti-symmetric tensor field, $F_{(3)}$, and the self-dual five-index tensor field, $F_{(5)}$, are given by

$$T_{MN}^{(1)} = P_M P_N^* + P_N P_M^*, \quad (3.2)$$

$$T_{MN}^{(3)} = \frac{1}{8}(G^{PQ}{}_M G_{PQ}^*{}_N + G^{*PQ}{}_M G_{PQ} - \frac{1}{6}g_{MN}G^{PQR}G_{PQR}^*), \quad (3.3)$$

$$T_{MN}^{(5)} = \frac{1}{6}F^{PQRS}{}_M F_{PQRS}{}_N. \quad (3.4)$$

We work here in the unitary gauge in which B is a complex scalar field and

$$P_M = f^2 \partial_M B, \quad Q_M = f^2 \text{Im}(B \partial_M B^*), \quad (3.5)$$

with

$$f = \frac{1}{(1 - BB^*)^{1/2}}, \quad (3.6)$$

while the antisymmetric tensor field $G_{(3)}$ is given by

$$G_{(3)} = f(F_{(3)} - BF_{(3)}^*). \quad (3.7)$$

- The Maxwell equations:

$$(\nabla^P - iQ^M)G_{MNP} = P^M G_{MNP}^* - \frac{2}{3}i F_{MNPQR}G^{PQR}. \quad (3.8)$$

- The dilaton equation:²

$$(\nabla^M - 2iQ^M)P_M = -\frac{1}{24}G^{PQR}G_{PQR}. \quad (3.9)$$

² A perceptive reader might have noticed that the sign on the right hand side of our dilaton equation is opposite to that in (4.11) and (5.1) of [15]. It appears that there is an error in [15] in passing from (4.10) to (4.11). An independent verification of the sign is to check whether the Bianchi identity $\nabla^M R_{MN} - \frac{1}{2}\nabla_N R = 0$ is consistent with the field equations. Indeed, this is the case for the sign in (3.9), but not for the one in [15]. The consistency between the Bianchi identities and the field equations was also verified in [17], with the resulting correct sign in the dilaton equation.

- The self-dual equation:

$$F_{(5)} = *F_{(5)}, \quad (3.10)$$

In addition, $F_{(3)}$ and $F_{(5)}$ satisfy Bianchi identities which follow from the definition of those field strengths in terms of their potentials:

$$\begin{aligned} F_{(3)} &= dA_{(2)}, \\ F_{(5)} &= dA_{(4)} - \frac{1}{8} \text{Im}(A_{(2)} \wedge F_{(3)}^*). \end{aligned} \quad (3.11)$$

Our strategy for constructing the ten-dimensional solution is to start with the consistent truncation Ansatz for the metric [8]. By examining the resulting Ricci tensor, we arrive at identities that, together with the $SU(2) \times U(1)^2$ symmetry, essentially determine the general structure of the antisymmetric tensor fields. The next crucial step is to solve the linearized Maxwell equation for the three index tensor field, $G_{(3)}$, which turns out to yield a non-vanishing source for the dilaton/axion field in (3.9). Using the consistent truncation Ansatz for the dilaton/axion we are then able to completely solve the Einstein equations to all orders, and fine tune all the phases and constants using the remaining equations of motion and the Bianchi identities.

3.1. The metric

Along the flow the ten-dimensional space-time is topologically a product of AdS_5 and a sphere, S_5 , with the “warped product” metric of the form:

$$ds_{10}^2 = \Omega^2 ds_{1,4}^2 + ds_5^2. \quad (3.12)$$

where $ds_{1,4}^2$ has been given in (2.4). The “internal” metric, ds_5^2 , and the warp factor, Ω^2 , are determined by the consistent truncation in terms of the scalar fields as discussed in the appendix.

The calculation of the explicit form of the internal metric using (A.8) is essentially the same as in [9] (see also [18]). We represent S_5 as a unit sphere in \mathbb{R}^6 with the cartesian coordinates x^I , $I = 1, \dots, 6$, and pass to suitable spherical coordinates to make the $SU(2) \times U(1)^2$ symmetry manifest. This is accomplished by setting

$$u^1 = x^1 + ix^4, \quad u^2 = x^2 + ix^3, \quad u^3 = x^5 - ix^6,$$

so that (u^1, u^2) transform as a doublet of $SU(2)$ with zero charge, and u^3 is a singlet with charge 1. The remaining $U(1)$ rotates between the doublet and its conjugate. Then we use the group action to reparameterize these coordinates as follows:

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \cos \theta g(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u^3 = e^{i\phi} \sin \theta, \quad (3.13)$$

where $g(\alpha_1, \alpha_2, \alpha_3)$ is an $SU(2)$ matrix expressed in terms of your Euler angles.

Using explicit scalar 27-beins, $\tilde{\mathcal{V}}_{IJ}{}^{ab}$, for the flow, and parametrizing the Killing vectors in terms of the coordinates above we arrive at the final result for the internal metric:

$$ds_5^2 = \frac{a^2 (cX_1X_2)^{1/4}}{2\rho^3} \left(c^{-1} d\theta^2 + \rho^6 \cos^2 \theta \left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2 + \sigma_3^2}{X_1} \right) + \sin^2 \theta \frac{d\phi^2}{X_2} \right) \quad (3.14)$$

where, as in section 2, $c \equiv \cosh(2\chi)$ and $\rho \equiv e^\alpha$. The warp factor is given by:

$$\Omega^2 = \Delta^{-\frac{2}{3}} = \frac{(cX_1X_2)^{1/4}}{\rho}. \quad (3.15)$$

The two functions, X_1 and X_2 , are defined by

$$\begin{aligned} X_1(r, \theta) &= \cos^2 \theta + \rho(r)^6 \cosh(2\chi(r)) \sin^2 \theta, \\ X_2(r, \theta) &= \cosh(2\chi(r)) \cos^2 \theta + \rho(r)^6 \sin^2 \theta. \end{aligned} \quad (3.16)$$

and we have introduced the constant, a , to account for the arbitrary normalization of the Killing vectors. As usual, σ_i , $i = 1, 2, 3$, are the $SU(2)$ left-invariant forms, satisfying $d\sigma_i = 2\sigma_j \wedge \sigma_k$.

Clearly, the metric (3.14) is invariant under $SU(2)$ and the two $U(1)$'s, where the first one, $U_\phi(1)$, acts by a translation in ϕ , while the second one, $U_{23}(1)$, rotates σ_2 into σ_3 .

Note that in our coordinates the metric (3.12) is almost diagonal. We choose the corresponding orthonormal frames e^M , $M = 1, \dots, 10$,

$$\begin{aligned} e^1 &\propto dx^0, & e^2 &\propto dx^1, & e^3 &\propto dx^2, & e^4 &\propto dx^3, & e^5 &\propto dr, \\ e^6 &\propto d\theta, & e^7 &\propto \sigma_1, & e^8 &\propto \sigma_2, & e^9 &\propto \sigma_3, & e^{10} &\propto d\phi. \end{aligned} \quad (3.17)$$

The computation of the Ricci tensor becomes rather involved and is most conveniently carried out on a computer. We find that the only non-vanishing off-diagonal components are $R_{56} = R_{65}$, while the diagonal components satisfy the obvious identities, $R_{11} = -R_{22} =$

$-R_{33} = -R_{44}$ and $R_{88} = R_{99}$, which follow from the symmetries of the metric. We also find the rather non-trivial identity:

$$R_{77} + R_{88} = 2R_{11}. \quad (3.18)$$

We will see below that this equation implies the vanishing of some components of the 3-form field strengths.

Throughout the calculation we use the flow equations (2.5) to eliminate derivatives with respect to r , so that the final result depends on rational functions of the scalar fields $c(r)$ and $\rho(r)$ and trigonometric functions of θ . While most of the components of the Ricci tensor are sufficiently complicated to prevent us from reproducing them here, we note that the combination $R_{10\,10} - R_{77}$ is rather simple. This will turn out important for solving the Einstein equations below.

3.2. The dilaton

We now use (1.2) to obtain the $SL(2, \mathbb{R}) \equiv SU(1, 1)$ scalar matrix of the ten-dimensional type IIB theory. We find that on the (α, χ) parameter space considered in section 2, the right-hand side of (1.2) yields:

$$\frac{1}{\rho^2} \begin{pmatrix} cX_1 \cos^2 \phi + X_2 \sin^2 \phi & \rho^6 s \sin^2 \theta \sin \phi \cos \phi \\ \rho^6 s \sin^2 \theta \sin \phi \cos \phi & cX_1 \sin^2 \phi + X_2 \cos^2 \phi \end{pmatrix}, \quad (3.19)$$

where $c = \cosh(2\chi)$ and $s = \sinh(2\chi)$. According to (1.2) the determinant should yield $\Delta^{-8/3}$, and from this we obtain:

$$\Delta = \frac{\rho^{3/2}}{(cX_1X_2)^{-3/8}}, \quad (3.20)$$

which is consistent with (3.15). Note that $\Delta^2 \equiv \det(g_{mp} \overset{\circ}{g}{}^{pq})$ where g_{mp} is the internal metric given by (3.14) and $\overset{\circ}{g}{}^{pq}$ is the inverse of the “round” internal metric at $\chi = \alpha = 0$. One can determine S in the symmetric gauge by taking the square root of this matrix. For direct comparison with the IIB field equations in [15] we also pass to the $SU(1, 1)$ basis in which the dilaton/axion matrix takes the form:

$$V = f \begin{pmatrix} 1 & B \\ B^* & 1 \end{pmatrix}. \quad (3.21)$$

We then find the following result:

$$f = \frac{1}{2} \left(\left(\frac{cX_1}{X_2} \right)^{\frac{1}{4}} + \left(\frac{cX_1}{X_2} \right)^{-\frac{1}{4}} \right), \quad fB = \frac{1}{2} \left(\left(\frac{cX_1}{X_2} \right)^{\frac{1}{4}} - \left(\frac{cX_1}{X_2} \right)^{-\frac{1}{4}} \right) e^{2i\phi}. \quad (3.22)$$

3.3. The antisymmetric tensor fields

A Poincare and $SU(2) \times U(1)^2$ invariant Ansatz for the self-dual antisymmetric tensor field, $F_{(5)}$, reads

$$F_{(5)} = \mathcal{F} + *\mathcal{F}, \quad \mathcal{F} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dw \quad (3.23)$$

where $w(r, \theta)$ is an arbitrary function. The self-duality equation (3.10) is then satisfied by construction. The non-vanishing components of the energy momentum tensor satisfy:

$$T_{11}^{(5)} = -T_{22}^{(5)} = \dots = -T_{33}^{(5)} = T_{77}^{(5)} = \dots = T_{10\,10}^{(5)} = \mathcal{A}^2 + \mathcal{B}^2, \quad (3.24)$$

$$T_{55}^{(5)} = -T_{66}^{(5)} = \mathcal{A}^2 - \mathcal{B}^2, \quad (3.25)$$

and

$$T_{56}^{(5)} = T_{65}^{(5)} = 2\mathcal{A}\mathcal{B}, \quad (3.26)$$

where

$$\begin{aligned} \mathcal{A} &= \frac{2^{3/2}}{ak^4} \frac{s^4}{c^{1/8} \rho^{9/2} (X_1 X_2)^{5/8}} \frac{\partial w}{\partial \theta}, \\ \mathcal{B} &= \frac{2}{k^4} \frac{s^4}{\rho^{11/2} (cX_1 X_2)^{5/8}} \frac{\partial w}{\partial r}, \end{aligned} \quad (3.27)$$

and k is the constant introduced in (2.10).

The most general Ansatz for the potential, $A_{(2)}$, that gives an $SU(2) \times U_{23}(1)$ invariant field strength, $G_{(3)}$, with the $U_\phi(1)$ charge +1 is

$$A_{(2)} = e^{i\phi} (a_1(r, \theta) d\theta \wedge \sigma_1 + a_2(r, \theta) \sigma_2 \wedge \sigma_3 + a_3(r, \theta) \sigma_1 \wedge d\phi + a_4(r, \theta) d\theta \wedge d\phi), \quad (3.28)$$

where $a_i(r, \theta)$ are some arbitrary complex functions.

In solving the equations of motion we want to test the conjecture (3.22), and assume as little as possible of its form. However, the group theory implies that the dilaton must have the form:

$$B = b(r, \theta) e^{2i\phi}. \quad (3.29)$$

In particular, the energy-momentum tensor of the dilaton vanishes in the directions 7, 8, 9 as well as in the directions parallel to brane. It follows that the only energy-momentum

tensor that can contribute to (3.18) is $T^{(3)}$. One finds that the only way to satisfy (3.1) and (3.18) is to impose:

$$a_4(r, \theta) = 0. \quad (3.30)$$

If one computes the components of the energy momentum tensors, $T_{MN}^{(1)}$ and $T_{MN}^{(3)}$, in the orthonormal frame (3.17) we find that they have some non-zero off-diagonal components where the corresponding components of the Ricci tensor vanish. These algebraic constraints are most simply (but not necessarily uniquely) solved by requiring that the phase of b (if any) is independent of r and θ , and the functions a_1 and a_2 are pure imaginary, while a_3 is real.

At this point it is instructive to solve the linearized equations of motion in the UV limit, $r \rightarrow \infty$. In this limit the metric (3.12) approaches the product metric on $AdS_5 \times S_5$ with the radii $L = 2/g$ and $a/\sqrt{2}$, respectively, as seen from the expansions $A(r) \sim r/L$, $\chi \sim e^{-r/L}$, $\alpha \sim re^{-2r/L}$ that follow from (2.5) and (2.6). The five-index tensor should reduce to the usual Freund-Rubin Ansatz. Thus we take

$$w = m \frac{k^4}{4s^4} + O\left(\frac{1}{s^2}\right) \quad (3.31)$$

and find that the UV limit reproduces the correct solution of the Einstein equations at the $\mathcal{N} = 8$ critical point provided

$$a^2 = \frac{8}{g^2} \quad \text{and} \quad m^2 = \frac{1}{16}. \quad (3.32)$$

Futhermore we consider a linearized Ansatz for the two-index gauge potentials of the form:

$$a_i(r, \theta) = e^{-\mu r/L} \tilde{a}_i(\theta) + O(e^{-(\mu+1)r/L}), \quad (3.33)$$

for some constant μ . From the linearized analysis of [2,3], and because these linearized fields are dual to fermion bilinears on the brane, we must have either $\mu = 1$ or $\mu = 3$. The former mode is non-normalizable and corresponds to a massive flow, while the latter is normalizable and corresponds to a “gaugino condensate,” which is a vacuum modulus.

The Maxwell equations do indeed imply that $\mu = 1$ or $\mu = 3$. There are also four independent Maxwell equations for the functions \tilde{a}_i , which can be reduced to a single third order equation (which has regular singular points) for $\tilde{a}_1(\theta)$. The regular solution is

$$\tilde{a}_1(\theta) = a_0 i \cos(\theta), \quad (3.34)$$

where a_0 is a real constant. The remaining two functions are

$$\tilde{a}_2(\theta) = a_0 i \cos^2 \theta \sin \theta \quad \text{and} \quad \tilde{a}_3(\theta) = -a_0 \cos^2 \theta \sin \theta. \quad (3.35)$$

Substituting this solution into the right hand side of the dilaton equation we find

$$\frac{1}{24} G_{MNP} G^{MNP} \propto (\mu^2 - 9) e^{-2\mu r/L} e^{2i\phi} \sin^2 \theta. \quad (3.36)$$

This reveals two interesting features: If $\mu = 3$ then the dilaton does *not* flow at lowest order, whereas if $\mu = 1$ then the dilaton *must* flow. So deforming the ground state of the Yang-Mills theory does not (at lowest order) require the dilaton to run, but if the flow involves giving a mass to the fermions then the dilaton must run.

4. The complete solution in ten dimensions

With the general structure of the solution determined by group theory and the linearized form, we now turn to the full solution. We take the metric in (3.12), (2.4) and (3.14), the dilaton in (3.21) and (3.22), the five-index tensor field, $F_{(5)}$, of the form (3.23) and the three-index tensor, $G_{(3)}$, defined by the potential $A_{(2)}$ in (3.28) with pure imaginary functions a_1 and a_2 , a real function a_3 and $a_4 = 0$.

4.1. Solving the equations of motion

We start with the Einstein equations (3.1) and consider linear combinations for which there are some cancellations or obvious simplifications of the energy momentum tensors on the right hand side.

The first such example is the difference between the (10,10) and (1,1) equations, where we find

$$R_{10\,10} - R_{11} \neq 0. \quad (4.1)$$

Now, it is easy to see that the reality conditions on the functions a_i imply the identity

$$T_{10\,10}^{(3)} - T_{11}^{(3)} = \frac{e^{-2i\phi}}{24} G_{MNP} G^{MNP}. \quad (4.2)$$

Combined with the dilaton equation (3.9), this provides us with a nontrivial test of the consistent truncation Ansatz for the metric and the dilaton, namely

$$R_{10\,10} - R_{11} = 2|P_{10}|^2 + e^{-2i\phi} (\nabla^M - 2iQ^M) P_M. \quad (4.3)$$

We find that this identity is indeed satisfied by the dilaton/axion given in (3.22).

In fact, the above calculation can be viewed as an independent confirmation that the ten-dimensional dilaton must run. The only input that goes into (4.2) is a symmetry constraint on which components, G_{MNP} , can be non-zero and that those components are purely real or imaginary up to the $e^{i\phi}$ phase. Thus, if we tried to set the dilaton to zero, we would end up with the vanishing right hand side in (4.3), which would contradict (4.1).

The next combination of the Einstein equations is the difference

$$R_{10\,10} - R_{77} - 2|P_{10}|^2 = T_{10\,10}^{(3)} - T_{77}^{(3)}, \quad (4.4)$$

where we have used $T_{77}^{(1)} = 0$ and $T_{10\,10}^{(5)} = T_{77}^{(5)}$. Evaluating (4.4) explicitly we get

$$\frac{g^2}{4} \frac{\rho^{-3} \sinh^2(2\chi) (2X_1 + \rho^6 X_2)^2}{(\cosh(2\chi) X_1 X_2)^{5/4}} - \frac{g^2}{4} \frac{\rho^9 \tanh^2(2\chi)}{(\cosh(2\chi) X_1 X_2)^{5/4}} = |G_{8910}|^2 - |G_{567}|^2, \quad (4.5)$$

and verify that in the linearized limit the first and the second term on the left hand side reduce to the first and the second term, respectively, on the right hand side. Thus we set

$$\begin{aligned} G_{567} &= i \frac{g}{2} \frac{\rho^{9/2} \tanh(2\chi)}{(\cosh(2\chi) X_1 X_2)^{1/8}} e^{i\phi}, \\ G_{8910} &= -\frac{g}{2} \frac{\sinh(2\chi) (2X_1 + \rho^6 X_2)}{\rho^{3/2} (\cosh(2\chi) X_1 X_2)^{5/8}} e^{i\phi}, \end{aligned} \quad (4.6)$$

where the signs have been chosen to agree with the linearized solution to the Maxwell equations.

By expanding (3.7) in terms of the potential and using (3.22) one obtains

$$\begin{aligned} G_{567} &= i \frac{g^2}{4} e^{i\phi} \frac{\rho^{1/2} c^{3/4} \sec \theta}{(c X_1 X_2)^{1/8}} \frac{\partial a_1}{\partial r}, \\ G_{8910} &= -\frac{g^3}{8} e^{i\phi} \frac{X_1 X_2 \csc \theta \sec^2 \theta}{\rho^{3/2} (c X_1 X_2)^{5/8}} (a_2 - 2a_3). \end{aligned} \quad (4.7)$$

The first order equation for $a_1(r, \theta)$ can be easily integrated, and by invoking once more the linearized limit we can identify the remaining two functions $a_2(r, \theta)$ and $a_3(r, \theta)$. The result is a simple modification of the linearized solution (3.34) and (3.35):

$$\begin{aligned} a_1(r, \theta) &= -i \frac{4}{g^2} \tanh(2\chi) \cos \theta, \\ a_2(r, \theta) &= i \frac{4}{g^2} \frac{\rho^6 \sinh(2\chi)}{X_1} \sin \theta \cos^2 \theta, \\ a_3(r, \theta) &= \frac{4}{g^2} \frac{\sinh(2\chi)}{X_2} \sin \theta \cos^2 \theta. \end{aligned} \quad (4.8)$$

At this stage a nontrivial check for our solution is the sum of the (5,5) and (6,6) Einstein equations in which according to (3.25) the contribution from the yet undetermined energy momentum tensor of the five-index tensor cancels.

Now, the five-index tensor is calculated using, e.g., the (1,1) and (5,5) Einstein equations and fixing the sign to agree with the linearized Ansatz. The result is

$$w(r, \theta) = \frac{k^4}{4} \frac{\rho^6 X_1}{\sinh^2(2\chi)}. \quad (4.9)$$

Finally, it is a matter of a straightforward algebra to verify that all the remaining equations of motion and the Bianchi identities are satisfied! The conjecture (1.2) has thus passed a collection of non-trivial tests perfectly.

4.2. Asymptotic behaviour of the solution

As was noted in section 2, there are three distinct asymptotic flows:

- (i) $\gamma > 0$: $\alpha \sim \frac{2}{3}\chi + \frac{1}{6}\log(\frac{\gamma}{4})$ for large (positive) χ .
- (ii) $\gamma < 0$: $\alpha \rightarrow -\infty$, $\chi \rightarrow \text{const.}$
- (iii) $\gamma = 0$: $\alpha \sim -\frac{1}{3}\chi + \frac{1}{6}\log(\frac{4}{3})$ for large (positive) χ .

Here we will focus primarily on the last option as we believe it should exhibit the most interesting new behaviour. As discussed earlier, the $\gamma > 0$ flow is expected to be unphysical, while the $\gamma < 0$ flow will be akin to the Coulomb branch of the $\mathcal{N} = 4$ theory.

First, the $\gamma = 0$ is most interesting in that the dilaton depends upon θ in the asymptotic limit. For $\gamma < 0$ we find $B \rightarrow 0$ as $\alpha \rightarrow -\infty$, and the asymptotic dilaton/axion configuration is trivial. Indeed, it approaches the constant dilaton background of the $\mathcal{N} = 4$ UV fixed point. For $\gamma > 0$ we have $B \rightarrow e^{2i\phi}$ as $\chi \rightarrow \infty$ and the matrix S diverges in no matter what direction we approach the “core” of the solution. For $\gamma = 0$ we have:

$$B = \left(\frac{(1 + \frac{2}{3}\tan^2(\theta))^{\frac{1}{2}} - 1}{(1 + \frac{2}{3}\tan^2(\theta))^{\frac{1}{2}} + 1} \right) e^{2i\phi}. \quad (4.10)$$

Note that generically $|B| < 1$, except for $\theta = \pi/2$ for which one has $B = e^{2i\phi}$. So the transition from $\gamma < 0$ to $\gamma > 0$ can be thought of as moving from an asymptotically trivial dilaton, to a non-trivial dilaton matrix that is finite except on the “ring” $\theta = \pi/2$, and thence to a dilaton matrix that is asymptotically singular in all directions.

One should note that $\theta = \pi/2$ corresponds to setting the cartesian coordinates $x^1 = x^2 = x^3 = x^4 = 0$ on S_5 . The remaining non-trivial coordinates x^5 and x^6 thus define a ring, which is presumably the enhançon ring of [19]. The fact that the dilaton is asymptotically trivial for $\gamma < 0$, is singular for $\gamma > 0$, and exhibits the milder ring singularity for $\gamma = 0$ further supports the identification of the supergravity flows with the various field theory limits.

The Einstein metric behaves similarly. Setting $\gamma = 0$ we find that as $\chi \rightarrow \infty$ the vielbein behaves according to:

$$\begin{aligned} e^a &\sim 2\nu e^{-2\chi} dx^a, \quad a = 1, \dots, 4; & e^5 &\sim -3a \nu d\chi, & e^6 &\sim \sqrt{\frac{3}{2}} a \nu d\theta, \\ e^7 &\sim \frac{1}{2\sqrt{2}} a \nu e^{-2\chi} \sigma_1, & e^8 &\sim \frac{1}{2\sqrt{2}} a \nu (1 + \frac{2}{3} \tan^2(\theta))^{-\frac{1}{2}} \sigma_2, \\ e^9 &\sim \frac{1}{2\sqrt{2}} a \nu (1 + \frac{2}{3} \tan^2(\theta))^{-\frac{1}{2}} \sigma_3, & e^{10} &\sim \sqrt{\frac{3}{2}} a \nu \tan(\theta) d\phi, \end{aligned} \quad (4.11)$$

where

$$\nu \equiv \left(\frac{2}{3}\right)^{\frac{1}{4}} (\cos \theta)^{\frac{1}{2}} (1 + \frac{2}{3} \tan^2(\theta))^{\frac{1}{8}}. \quad (4.12)$$

As $\chi \rightarrow \infty$, and for $\theta \neq \pi/2$, the metric remains regular, and the $D3$ -branes appear to be at the bottom of an infinitely long throat, much as they are at a conformal fixed point. The metric does not quite have asymptotic conformal invariance owing to the χ -dependence of e^7 . It is however tempting to speculate that the “near-conformality” of this asymptotic metric may be related to a flow to the large N versions of Argyres–Douglas points [20,21].

At $\theta = \pi/2$ the metric (as well as the dilaton) become singular. One can re-analyse the asymptotics, and the precise details depend upon whether one looks at the Einstein metric or string metric. The latter still sees an infinitely long throat, whereas the Einstein metric is singular at finite distance. We also find that in either metric, the coefficient of $d\phi$, and hence the diameter of the “ring” singularity goes to *infinity* as $\chi \rightarrow \infty$.

5. Conclusions

In obtaining and checking (1.2), we have exposed a very interesting aspect of consistent truncation, whose physical consequence is that the five-dimensional “dilaton coset” should be identified with the $SL(2, \mathbb{Z})$ -invariant $\mathcal{N} = 4$ coupling, and not with coupling in the gauge theory on the brane. Indeed, our expression gives the coupling on the brane as a function of the $\mathcal{N} = 4$ coupling and of the masses and vevs captured by gauged $\mathcal{N} = 8$ supergravity.

Many of the the flows obtained in the literature to date [6,7,13,11,22,23,24] keep the five-dimensional dilaton fixed. For these flows (1.2) yields the flow of the dilaton as a function the masses and vevs that drive that flow, and if the flow is supersymmetric (1.2) must capture the NSVZ exact beta function. Indeed, it is, in hindsight rather easy to identify the IIB supergravity version of the NSVZ exact beta function: The supersymmetry variation of the ten-dimensional fermion is [15]:

$$\delta\lambda = \frac{i}{\kappa} \gamma^\mu P_\mu \epsilon^* - \frac{i}{24} G_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon. \quad (5.1)$$

This must vanish along supersymmetric flows. To linear order $G_{\mu\nu\rho}$ are the fermion masses, while P_r is the running coupling.

The foregoing identification is, however, a little superficial in that it glosses over the fact the dilaton doesn't just depend upon the radius, r . It also depends upon other coordinates. For the $\mathcal{N} = 2$ flow considered here it depends upon θ and ϕ (the latter being very simple). This makes physical sense in that one starts with an $\mathcal{N} = 4$ theory, and the flow must “know” which directions are “getting massive.” To be more specific, Seiberg and Witten [25] showed that there was no infra-red gauge enhancement on the Coulomb branch of $\mathcal{N} = 2$ theories. Thus, even in the far infra-red, the brane description of the $\mathcal{N} = 2$ supersymmetric limit must involve a “disk-like” distribution of branes, or at least something with a corresponding multipole moment. Thus it is entirely to be expected that the metric and dilaton have non-trivial dependence on θ and ϕ in the IR limit. This does, however, beg the question as to how the direction of approach is seen purely from the perspective of the brane and, in particular, in terms of the Seiberg-Witten effective action. More generally, for $\mathcal{N} = 1$ flows, in which several fields are given independent masses, one would like to relate the direction on S_5 with the physics on the brane and thereby isolate the running coupling of [12].

As yet we do not have definitive answers to these issues, but we have computed some of the dilaton flows for other known supersymmetric flows. First, and rather surprisingly, the dilaton and axion are constant everywhere on the two-parameter (α, χ) space underlying the flow of [13]. On the other hand, the dilaton and axion *do* flow in a very non-trivial way for the flow of [22]. This is presently under investigation.

Finally, and rather ironically, prior to this work we expended much effort in trying to find supersymmetric flows in which the five-dimensional dilaton flows along with other fields. Our failed efforts, and a heuristic argument based upon energy suggest that there should be a no-go theorem for such supersymmetric flows. It would be interesting to try

to prove such a result, and then (1.2) would truly be the unique expression for the running coupling on the brane.

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Appendix A. Consistent truncation revisited

The central issue in consistent truncation is to determine how the fields and field equations of the lower-dimensional theory are embedded in those of a higher-dimensional theory. One of the keys to this is to use the gauge invariances of both theories and supersymmetry transformations to make this mapping precise [26]. In particular, this is how one can find the exact metric of the higher-dimensional theory from the metric and scalar fields of the low-dimensional theory.

The consistent truncations has been carried out in full detail for the reduction of the eleven dimensional supergravity to four dimensions (see, e.g., [26,18,27] and the references therein) and to seven dimensions [28,29]. The consistent truncation of IIB supergravity has been analyzed only for various subsectors of the theory (see, e.g., [8,18,29]). Here we briefly review this technique, and then extend it to the consideration of the dilaton and axion in IIB supergravity.

A.1. The exact form of the internal metric

The starting point is the encoding of the gauge fields of the dimensionally reduced theory into the metric and Killing vectors of the parent theory. While this technique had always been a staple of dimensional reduction at the linearized order, it was argued in [30] that when properly stated, such encoding of gauge fields must be exact to all orders in fields. The argument was based upon how the gauge symmetries in the reduced theory must be related to the diffeomorphism invariance of the parent theory, and that this relationship would be spoilt if the linearized Ansatz were not, in fact, exact.

To be more specific, consider a theory in D -dimensions that is reduced to d -dimensions on a manifold, \mathcal{M} , with isometries represented by Killing fields: $K^{\Omega p}$, where Ω indexes

the Killing fields and p is the vector index on \mathcal{M} . Decompose the D -dimensional vielbein according to:

$$e_M^A = \begin{pmatrix} e_\mu^\alpha & e_\mu^a \\ e_m^\alpha & e_m^a \end{pmatrix}, \quad (\text{A.1})$$

where the upper left corner represents the d -dimensional space-time and the bottom right represents the $(D - d)$ -dimensional manifold \mathcal{M} . The claim of [30] is that

$$e_\mu^a = A_\mu^\Omega K^{\Omega p} e_p^a \quad (\text{A.2})$$

is the exact consistent truncation Ansatz for the gauge fields to all orders.

It was observed in [26] that when the foregoing is combined with the supersymmetry transformations of the gauge fields, one could obtain the exact Ansatz for the internal metric g_{pq} . To illustrate this we consider the S_5 reduction of the IIB supergravity to five dimensions, but we stress that the argument is very general. Consider the gravitino terms in the supersymmetry variations of the five-dimensional gauge vector fields [4,5]:

$$\delta A_\mu^{IJ} = 2i \tilde{\mathcal{V}}_{IJab} \tilde{\epsilon}^a \hat{\psi}_\mu^b + \dots \quad (\text{A.3})$$

In this equation, $I, J = 1, \dots, 6$, and $A^{IJ} = -A^{JI}$ represent the $SO(6)$ gauge fields. The hats $\hat{}$ have been introduced to distinguish five-dimensional fields from their ten-dimensional antecedents.

Using (A.2) the corresponding ten-dimensional variation of the vielbein gives:

$$\delta A_\mu^{IJ} K^{IJp} = (\delta e_\mu^a) e_a^p + e_\mu^a (\delta e_a^p) = -2\kappa \text{Im}(\tilde{\epsilon} \gamma^a \psi_\mu) + \dots \quad (\text{A.4})$$

One now recalls that the dimensional reduction to the standard Einstein action and Rarita-Schwinger actions in d dimensions requires a proliferation of “warp” factors. In particular, one has:

$$e_\mu^\alpha = \Delta^{-\frac{1}{d-2}} \hat{e}_\mu^\alpha; \quad \psi_\mu = \Delta^{-\frac{1}{2(d-2)}} \hat{\psi}_\mu; \quad \epsilon = \Delta^{-\frac{1}{2(d-2)}} \hat{\epsilon}, \quad (\text{A.5})$$

where the hats refer to d -dimensional quantities, and the warp-factor is given by:

$$\Delta \equiv \det(e_p^a \circ e_b^p) = \sqrt{\det(g_{mp} \circ g^{pq})}. \quad (\text{A.6})$$

The inverse frame $\circ e_b^p$ and the inverse metric $\circ g^{pq}$ are those of the “round,” maximally supersymmetric background on \mathcal{M} . The metric warp-factor is introduced so that the D -dimensional Einstein action, along with its factors of \sqrt{g} reduce to the d -dimensional

Einstein action. The Rarita-Schwinger field is rescaled for the same reason, and the supersymmetry parameter is rescaled so that one gets the canonical form for the supersymmetry variations of $\widehat{\psi}_\mu$ and $\widehat{e}_\mu{}^\alpha$ in d -dimensions. The effect of all this warping is that (A.4) becomes:

$$\delta A_\mu^{IJ} K^{IJp} = -2\kappa e_a{}^p \Delta^{-\frac{1}{d-2}} \text{Im}(\bar{\epsilon}\gamma^a \widehat{\psi}_\mu) + \dots \quad (\text{A.7})$$

One now compares (A.3) with (A.7). The internal indices of the former, arise through the labelling of Killing spinors \mathcal{M} . For the five-dimensional supergravity this labelling is ambiguous up to a $USp(8)$ transformation,³ but such internal local symmetry transformations can be eliminated by squaring and contracting with the $USp(8)$ symplectic form: Ω^{ab} . The result is the squaring and contraction of the inverse-vielbein $e_a{}^p$ in (A.7) yields the inverse metric, and one obtains (putting $d = 5$):

$$\Delta^{-\frac{2}{3}} \widehat{g}^{pq} = \frac{1}{a^2} K^{IJp} K^{KLq} \widetilde{\mathcal{V}}_{IJab} \widetilde{\mathcal{V}}_{KLcd} \Omega^{ac} \Omega^{bd} \,, \quad (\text{A.8})$$

where the constant, a , depends upon the normalization of the Killing vectors.

This argument was performed in great detail in [26] for the S_7 reduction of eleven-dimensional supergravity. The foregoing argument was also made to arrive at the expression for the general IIB compactification metric given in [8]. The details of the local $USp(8)$ structure in ten dimensions were not explicitly checked for the result in [8], and so for that reason we referred to it as a ‘‘conjecture,’’ however, a more precise statement of that result would have been: If the truncation is consistent, then the internal metric must be given by (A.8).

A.2. The exact form for the dilaton

In the same spirit as in [8], we will now conjecture an exact form for the ten-dimensional dilaton.

The starting point is now the encoding of the five-dimensional tensor gauge fields in the ten-dimensional, two-form gauge potentials A_{MN}^α . At the linearized order one has:

$$A_{\mu\nu}^\alpha = \widehat{B}_{\mu\nu}^{I\alpha} x^I \,, \quad (\text{A.9})$$

where $\alpha = 1, 2$ now denotes an $SL(2, \mathbb{R})$ index, I is the $SO(6)$ vector index, and x^I are the cartesian coordinates of the 5-sphere: $\sum_I (x^I)^2 = 1$.

³ Recently, equations that determine those $USp(8)$ transformations were obtained in [29].

It seems plausible, based on the gauge symmetries, the minimal couplings, and mixings with the gauge fields, that this linearized Ansatz is exact to all orders. Rather than prove this in detail, we shall assume that it is true and derive the dilaton Ansatz. The body of this paper then represents a highly non-trivial test of this assumption.

One proceeds exactly as in the previous subsection, except that one compares specific gravitino terms in the ten-dimensional and five-dimensional supersymmetry variations of the two-form field strengths:

$$\begin{aligned}\delta A_{\mu\nu}^{\alpha} &= 4i V_+^{\alpha} \bar{\epsilon} \gamma_{[\mu} \psi_{\nu]}^* + 4i V_-^{\alpha} \bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} + \dots \\ &= \Delta^{-\frac{2}{3}} \left(4i V_+^{\alpha} \hat{\bar{\epsilon}} \gamma_{[\mu} \hat{\psi}_{\nu]}^* + 4i V_-^{\alpha} \hat{\bar{\epsilon}}^* \gamma_{[\mu} \hat{\psi}_{\nu]} \right) + \dots ,\end{aligned}\tag{A.10}$$

and

$$\delta \widehat{B}_{\mu\nu}^{I\alpha} = 2ig \epsilon^{\alpha\beta} \mathcal{V}_{I\beta ab} \bar{\widehat{\psi}}_{[\mu}^a \gamma_{\nu]} \epsilon^b + \dots .\tag{A.11}$$

Again one can remove the local $USp(8)$ transformations by squaring and contracting to obtain:

$$\Delta^{-\frac{4}{3}} (S S^T)^{\alpha\beta} = \text{const} \times \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \mathcal{V}_{I\gamma}{}^{ab} \mathcal{V}_{J\delta}{}^{cd} x^I x^J \Omega_{ac} \Omega_{bd} ,\tag{A.12}$$

where S^{α}_{β} is the IIB dilaton/axion matrix written in the $SL(2, \mathbb{R})$ basis, with the local $U(1) = O(2)$ acting from the right. (Note that V_{\pm}^{α} of [15] is generally written in the $SU(1, 1)$ basis.)

One obvious consistency check is that the value of Δ obtained from taking the determinants of both sides of (A.12) and of (A.8) agree. This was indeed confirmed in section 3.2.

One can choose a gauge in which $S \in SL(2, \mathbb{R})$ is symmetric, and so one can take the square root of (A.12), and extract S . One should also note that if the $E_{6(6)}$ matrix \mathcal{V} of five-dimensional scalars is, in fact, in the subgroup $SL(6, \mathbb{R}) \times SL(2, \mathbb{R})$ used in [4], then the IIB dilaton and axion are indeed precisely described by the $SL(2, \mathbb{R})$ factor of this $SL(6, \mathbb{R}) \times SL(2, \mathbb{R})$. The formula (A.12) is consistent with this special case. However, for general the $E_{6(6)}$ matrix \mathcal{V} , the five-dimensional and ten-dimensional $SL(2, \mathbb{R})$ factors have a highly non-trivial relationship.

In terms of the AdS/CFT correspondence, this last statement means that on the Coulomb branch of the $\mathcal{N} = 4$ Yang-Mills theory, the gauge coupling and theta angle are constant, and are represented by the $SL(2, \mathbb{R})$ of the five-dimensional theory. However, if fermion masses are turned on in the gauge theory, then (A.12) tells us exactly how the running of the gauge coupling and axion are determined entirely in terms of the running of all the masses of the Yang-Mills scalar and fermion fields. The $SL(2, R)$ matrix of the

five-dimensional theory is a global symmetry of the potential and is presumably broken to $SL(2, \mathbb{Z})$ in the quantum theory: it represents the symmetry of even the perturbed theory under the $SL(2, \mathbb{Z})$ action on the $\mathcal{N} = 4$ coupling. The actual physical coupling, represented by the ten-dimensional dilaton and axion, are non-trivial functions of this $\mathcal{N} = 4$ coupling, the masses of the fields and the scale. This relationship is given in large N theories by (A.12).

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